THE BLACK-SCHOLES MODEL FOR EUROPEAN OPTIONS ON TWO ASSETS

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Contents

CHAPTER 1

Introduction

Since the very first appearance in 1973 in the paper *The Pricing of Options and Corporate Liabilities* [**1**] by Fischer Black and Myron Scholes, option pricing has been one of the key parts of the financial world. Nowadays there exist several kinds of financial derivatives and more types of options than the European. Most of them already have a pricing method which, following the lines of the Black-Scholes model, is derived by canceling the risk and therefore obtaining a partial differential equation for the value of the option. This equation, under given conditions, may be turned into the heat equation by applying the appropriate change of variables. The heat equation is one of the most well-known second order PDEs, also known as one of the equations of the mathematical physics. This equation has very well-defined properties, since it admits a fundamental solution with which we can recover the solution to the initial value problem by doing the convolution of the initial condition with the fundamental solution.

Even though *a priori* it may seem that we are completely on the field of PDEs, it is not completely true. In order to derive the Black-Scholes equation one requires of tools provided by the stochastic calculus, particularly, a very important result analogous to the Taylor theorem for functions which depend not only on deterministic variables, but also on stochastic ones, more specifically, Itô processes. By using this result, known in the financial world as Itô's Lemma, one can construct a portfolio in which the randomness can be canceled. There are generally two standard methods to do it. The purely stochastic way, or martingale approximation, that proceeds by creating a portfolio whose value in the expiry date coincides with the value of the option. This yields a formula to evaluate the value of the option by calculating an expected value. Then, there is also the PDEs approach, known as hedging method, that consists in creating a portfolio which contains a given quantity of assets and options, in such a way that one eliminates the risk of the portfolio. Since this project is PDEs-oriented, all results will be obtained by using the second method, which it also turns out to be more intuitive and easy-going than the first one.

We have started talking about European options and we have mentioned that there are other kinds, such American, Asiatic or Exotic options. In this project we will treat mainly the first type, that is, European options. However, the difficulty will not be on the kind of option but on the assets the options lay on. The main aim, in fact, will be to determine the PDE that an option on two assets needs to fulfill. These two assets may be correlated. In the Chapter 2, we will introduce the necessary concepts of stochastic calculus, a fundamental tool in the development of the project. In the Chapter 3, we will start with the theory for European options and in the same chapter derive a few results for options which actually do not differ that much conceptually speaking from the European type, but they do have a particular interest by being on two assets. Next in Chapter 4, from a theoretical point of view, that is, that it may have not much relevance from a financial point of view, we will study two pathological cases. The first one will simply be to see what happens when one of the assets is in fact zero and see if we recover the case of one single variable. Secondly we will see what happens when the stochastic components of the assets are totally correlated. Finally in Chapter 5, we will solve numerically a few PDEs we have obtained by using the finite differences method in order to picture the results. Since the interest does not lay on the complexity of the problem but on its numerical solution, the programs will be basics in the structural sense, that is, they will be simple and specifics to solve the problem.

Regarding to the section of stochastic calculus, Chapter 2, since it is not the main field of the project, we will not give any proof of the results we state and we will not go into so much detail. The definitions and results may be nevertheless found in [**2**], [**4**], [**3**], [**5**] or [**12**]. Since this is a translation of the original project, proofs will not be given here since they are already in the original. Annexes will not be provided either.

CHAPTER 2

Previous concepts. Stochastic calculus

In this chapter we will give notions and results of stochastic calculus that we will use later on. Nevertheless, we will not prove any result since it is not the aim of the project. We will suppose that we have a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

DEFINITION 2.1. A filtration in a probability space is a family $\{\mathcal{F}_t, t \in \mathbb{T}\}$ of sub σ-algebras of $\mathcal{F}$ such that if $s \leq t$ then $\mathcal{F}_s \subseteq \mathcal{F}_t$.
A stochastic process $X = \{X(t), t \in \mathbb{T}\}$ is said to be adapted to the filtration $\{\mathcal{F}_t, t \in \mathbb{T}\}$ if $X(t)$ is $\mathcal{F}_t$ measurable $\forall t \in \mathbb{T}$. We denote $\mathcal{F}_t^X$ the σ-algebra generated by the stochastic variables $X(s)$, $s \leq t$.

From now onward we will consider that we work with a known filtration and that the adapted processes are defined according to the filtration defined above. The next definition makes reference to a concept widely used in the branch of probability and stochastic calculus. This is the background of what we will use, for instance, to model the asset price.

DEFINITION 2.2. A Brownian motion is a stochastic process $\{W(t), t \in (0, +\infty)\}$ such that

(1) $W(0)$=0 a.s.
(2) For each $0 \leq s \leq t$ the random variable $W(t) - W(s)$ is Gaussian with mean 0 and variance $t - s$. It will be denoted by $W(t) - W(s) \sim \mathcal{N}(0, t-s)$.
(3) For each $0 \leq t_1 < \ldots < t_r$ the random variables

$$W(t_1), W(t_2) - W(t_1), \cdots, W(t_r) - W(t_{r-1})$$

are independents.
(4) With probability 1, the paths

$$W(\cdot) : (0, +\infty) \longrightarrow \mathbb{R}$$

are continuous functions.

Now we give the natural generalized definition.

DEFINITION 2.3. A standard Brownian motion in $\mathbb{R}^n$ is a stochastic process $(\mathbb{Z}(t))_{t \geq 0}$ that at time t is given by a vector of n independent Brownian motions

$$\mathbb{Z}(t) = (Z_1(t), \ldots, Z_n(t)).$$

The next lemma (if we exclude the condition on ρ_{ij}) is known as Cholesky decomposition, and is a result of linear algebra that may be found in any algebra handbook, as for instance [**6**].

LEMMA 2.1. *Let $\rho = (\rho_{ij})_{1\leq i,j\leq n}$ be a (constant) symmetric positive defined matrix such that $\rho_{ii} = 1$ and $-1 \leq \rho_{ij} \leq 1$. Then a unique lower (or upper) triangular matrix $\mathbb{H}$ exists such that*

$$\rho = \mathbb{H}\mathbb{H}^T.$$

Next we will prepare the background of what our study case will be, that means, we will construct a way to deal with correlated processes.

PROPOSITION 2.1. *Let us consider a standard n-dimensional Brownian $\mathbb{Z}(t)$. Let us also consider $\mathbb{H}$ and ρ from Lemma 2.1. Let us define a new vector-process*

$$\mathbb{W}(t) = \mathbb{H}\mathbb{Z}(t). \tag{1}$$

This process has the following properties

(1) $\mathbb{W}(0) = 0$.
(2) *If $s \leq t$, then $\mathbb{W}(t) - \mathbb{W}(s)$ is multivariate normal, with mean 0 and variance-covariance matrix $(t-s)\rho$, i.e.*

$$\mathbb{W}(t) - \mathbb{W}(s) \sim \mathcal{N}\left(0, (t-s)\rho\right).$$

(3) *If $0 \leq r \leq s < t$ then the random variables $\mathbb{W}(t) - \mathbb{W}(s)$ and $\mathbb{W}(r)$ are independents, i.e. each component of the later is independent from the former.*
(4) *The paths or realizations $t \longrightarrow \mathbb{W}(t)$ are continuous with probability 1.*

The process $\mathbb{W}(t)$ will be called correlated Brownian motion.

The following result is part of a rigorous construction which requires of the martingale concept and Itô integrals in order to manipulate these quantities as desired. However, since developing such a thing would take a lot of time and effort and it is not the aim of the project, we will announce the lemma that will let us deal with Itô's differential notation.

LEMMA 2.2. *Let $(\mathbb{Z}(t))_{t\geq 0}$ be a standard Brownian motion. Itô's rules for the components $Z_i(t), Z_j(t)$ are given in differential notation by*

$$\mathrm{d}Z_i(t)\mathrm{d}Z_j(t) = \delta_{ij}\mathrm{d}t, \quad \mathrm{d}Z_i(t)\mathrm{d}t = 0, \quad \mathrm{d}t\mathrm{d}t = 0,$$

where δ_{ij} is the Kronecker delta.

Notice that for orders greater than 1 in dt all products of differential elements vanish.

COROLLARY 2.1. *More generally, if $(\mathbb{W}(t))_{t\geq 0}$ is a correlated Brownian motion with correlation matrix $\rho = \mathbb{H}\mathbb{H}^T$, then*

$$\mathrm{d}W_i(t) = \sum_{k=1}^{n} h_{ik}\mathrm{d}Z_k(t).$$

Consequently

$$\mathrm{d}W_i(t)\mathrm{d}W_j(t) = \rho_{ij}\mathrm{d}t.$$

Next we introduce the concept of Itô process.

Definition 2.4. A stochastic process $(\mathbb{X}(t))_{t\geq 0}$ is called vector Itô process if it is adapted and $\mathrm{d}\mathbb{X}(t) = (\mathrm{d}X_1(t), \ldots, \mathrm{d}X_n(t))$ is related to a m-dimensional standard Brownian motion $(\mathbb{Z}(t))_{t\geq 0}$ by

$$\mathrm{d}\mathbb{X}(t) = \mathbb{A}(t)\mathrm{d}t + \mathbb{H}(t)\mathrm{d}\mathbb{Z}(t), \tag{2}$$

where

$$\mathbb{A}(t) = \begin{pmatrix} a_1(t) \\ \vdots \\ a_n(t) \end{pmatrix}, \quad \mathbb{H}(t) = \begin{pmatrix} h_{11}(t) & \ldots & h_{1m}(t) \\ \vdots & \ddots & \vdots \\ h_{n1}(t) & \ldots & h_{nm}(t) \end{pmatrix}.$$

And each of the $a_k(t)$ and $h_{ij}(t)$ are functions of the time and random variables, fulfilling that

$$\int_0^T |a_k(t)|\,\mathrm{d}t < \infty \text{ a.s.}, \quad \int_0^T h_{ij}(t)^2\mathrm{d}t < \infty \text{ a.s.}$$

Let us observe that a correlated Brownian motion is a particular case of an Itô process for $\mathbb{A}(t) = 0$ i $\mathbb{H}(t)$ constant and squared. Finally, we introduce Itô's Lemma, one of the most important results that we will use.

Theorem 2.1 (Itô's Lemma). *Let $f : \Omega \times I \longrightarrow \mathbb{R}$ be with $\Omega \subset \mathbb{R}^n$ open and $I \subset \mathbb{R}$ closed interval or the positive real line. Let us suppose that $f = f(x,t) \in \mathcal{C}^{2,1}(\Omega \times I)$. Let $\mathbb{X}(t) = (X_1(t), \cdots, X_n(t))$ be a vector Itô process defined as in (2). Then*

$$\mathrm{d}f(\mathbb{X}(t), t) = f_t\mathrm{d}t + \sum_{k=1}^{n} f_{x_k}\mathrm{d}X_k(t) + \frac{1}{2}\sum_{i,j=1}^{n} f_{x_i x_j}\mathrm{d}X_i(t)\mathrm{d}X_j(t), \tag{3}$$

where the subscript in the function f denotes the partial derivative with respect the variable of the subscript. Moreover we assume all variable evaluated on the Itô process.

Next we are going to characterize the financial objects and structures from the point of view of stochastic calculus, by modeling things such as the value of an asset or a portfolio.

Definition 2.5. An asset is an economic resource. Anything tangible or intangible that is capable of being owned or controlled to produce value and that is held to have positive economic value is considered an asset.

Examples could be the shares of a company, bonds, currencies or commodities. When observing the value of an asset through the time, a way that comes up to model it, is by using an Itô process, considering two characteristic parameters for the active, its drift and volatility. A result which is a direct consequence of this model is the following one:

Proposition 2.2. *Let $(\mathbb{W}(t))_{t\geq 0}$ be a correlated Brownian motion defined in (1). Let us assume that a set of assets describes the Itô process induced by the Brownian motion,*

$$\mathrm{d}S_i(t) = \mu_i(t)S_i(t)\mathrm{d}t + \sigma_i(t)S_i(t)\mathrm{d}W_i(t),$$

where $\mu_i(t)$ is the drift of the asset and $\sigma_i(t)$ its volatility. Then the solution is given by

$$S_i(t) = S_i(0)\exp\left(\int_0^t \left(\mu_i(\tau) - \sigma_i^2(\tau)/2\right)\mathrm{d}\tau + \int_0^t \sigma_i(\tau)\mathrm{d}W_i(\tau)\right).$$

The proof of the proposition is based on results of stochastic calculus as well as the Itô's Lemma. Moreover, from now on, we will consider that for all t $\sigma_i > 0$. Next we will define the concept of portfolio as well as the type of portfolios we are interested in.

Definition 2.6. Let us consider the set of assets $S_i(t)$, being $S_0(t) = B(t) = \exp\left(\int_0^t r(\tau)\,\mathrm{d}\tau\right)$ the price of a bond so the currency earns interest with a ratio $r(t)$. Furthermore, let us assume that $S_i(t)$ are Itô processess as the ones defined as in Proposition 2.2. Let us consider $\theta_i(t)$ the amount of asset i we hold. A financial portfolio Θ is the stochastic process defined by

$$\Theta = \left\{(\theta_i(t))_{i=0}^n, t \in [0,T)\right\}.$$

And its value is

$$\Pi(t) = \Theta(t)\cdot S(t),$$

where $A \cdot B = \sum_{i=0}^n A_i B_i$ being $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$.

We will usually refer to the portfolio as its value $\Pi(t)$ instead of the holdings of the assets Θ.

Definition 2.7. Properties of a portfolio

(1) A portfolio $\Pi(t) = \theta(t)\cdot S(t)$ is called self-financed if $\mathrm{d}\Pi(t) = \theta(t)\cdot \mathrm{d}S(t)$.
(2) We will say that a portfolio Π with $\mathrm{d}\Pi(t) = u(t)\mathrm{d}t + v(t)\cdot\mathrm{d}\mathbb{W}(t)$ is risk-free if $v \equiv 0$.
(3) A portfolio is called to be hedging if it is self-financed and risk-free.

Notice that, by construction, a portfolio Π is an Itô process. Next we will give a definition that will be the base of our assumtions later on.

Definition 2.8. Financially speaking, an arbitrage is the chance of beginning with zero value, get a finite debt and end up without any loses and, with positive probability, with earnings.

The equivalent definition from the stochastic calculus point of view as we have modeled the portfolios is the following one.

Definition 2.9. An arbitrage is a self-financed portfolio Π defined in $[0,T]$, such that

(1) $\Pi(0) = 0$.
(2) $\exists M > -\infty$ such that for each t $\Pi(t) \geq M$ a.s.
(3) $\Pi(T) \geq 0$ a.s. and $\Pi(T) > 0$ with positive probability.

Finally, we will give a result that will let us characterize the risk-free portfolios in a market free of arbitrage.

PROPOSITION 2.3. *Let us assume that the market admits no arbitrage and that the currency earns interest with ratio* $r(t)$. *Let* Π *be a portfolio given by* $\mathrm{d}\Pi = u\mathrm{d}t + v \cdot \mathrm{d}\mathbb{W}$, *where* u *is continuous almost surely. Let us further assume that* Π *is hedging and almost surely bounded in* $\alpha \leq t \leq \beta$. *Then* $u(t) = r(t)\Pi(t)$ *almost surely. Equivalently,* $\mathrm{d}\Pi = r\Pi\mathrm{d}t$.

CHAPTER 3

Deduction by replication method

In this part we will use the replication method in order to provide the Black-Scholes equation for an option on two assets. This method consists in creating a portfolio Π so that it is always in a hedging position and which involves the value of the option as well as the asset's one. Then, by a simple arbitrage reason, and making the combination risk-free, we will conclude that the value of the option needs to fulfill the so-called Black-Scholes equation.

1. European options

Let us start defining how we will model the value of the two assets on which we will work through the project. These will have a deterministic drift μ, ν and also deterministic volatility σ_p, σ_q. Moreover, the Brownian motions will be correlated, with correlation coefficient ρ. As we have described in Proposition 2.1, these Brownian motions are of the form

$$\mathbb{W}(t) = \mathbb{H}\mathbb{Z}(t).$$

So if we take

$$\mathbb{H} = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}$$

it is trivial to see that

$$\mathbb{H}\mathbb{H}^T = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

Thus, we introduce the following result.

PROPOSITION 3.1. *Let us consider the Itô process given by*

$$\begin{cases} \mathrm{d}P = P\left(\mu \mathrm{d}t + \sigma_p \mathrm{d}X\right), \\ \mathrm{d}Q = Q\left(\nu \mathrm{d}t + \sigma_q\left(\rho \mathrm{d}X + \sqrt{1-\rho^2}\mathrm{d}Y\right)\right), \end{cases} \tag{4}$$

with $\sigma_{p,q}, \mu, \nu$ are deterministic functions of t and $-1 < \rho < 1$ constant, being X, Y independent Brownian motions. The the solution is given by

$$\begin{aligned} P =& P_0 \exp\left(\int_0^t \left(\mu(\tau) - \sigma_p^2(\tau)/2\right)\mathrm{d}\tau + \int_0^t \sigma_p(\tau)\mathrm{d}X(\tau)\right), \\ Q =& Q_0 \exp\left(\int_0^t \left(\nu(\tau) - \sigma_q^2(\tau)/2\right)\mathrm{d}\tau + \rho\int_0^t \sigma_q(\tau)\mathrm{d}X(\tau)\right. \\ & \left. + \sqrt{1-\rho^2}\int_0^t \sigma_q(\tau)\mathrm{d}Y(\tau)\right). \end{aligned}$$

Let us observe that the stochastic variables above mentioned are positives.

DEFINITION 3.1. A call European option is a contract with the following characteristic: at a given time in the future, known as expiry date, the holder can buy the aforementioned asset, known as underlying asset or simply underlying, for a given amount, known as exercise price or strike price.

Notice that for the holder of the option, the contract is a right, not an obligation. On the other hand, for the writer or seller, he is obligated to sell the asset if the holder chooses to buy it.

DEFINITION 3.2. A put European option is a contract that allows the holder to sell the asset for a given amount at a given time.

In general we will denote the call option by $\mathcal{C}$ and the put option by $\mathcal{P}$. However, when we are not making any reference to the type, we will just write V. Moreover, the expiry date will be denoted by T, and the exercise price by E. The pay off or value of the option at the expiry date, $V\left(p,q,T\right)$, will be denoted by $\Lambda\left(p,q\right)$ and from now on we will assume that it is non-negative for all $(p,q)\in\left(0,+\infty\right)^2$ and strictly positive on a set of positive measure.

THEOREM 3.1. *Let us assume that the value of the assets follows the Itô process given in (4). Let us also assume that the market admits no arbitrages and that the value of an European option is a differential function $V\left(p,q,t\right)$. Further, assume that the currency has a deterministic interest rate $r(t)$. Then V fulfills*

$$V_t+\frac{\sigma_p^2p^2}{2}V_{pp}+\rho\sigma_p\sigma_q pqV_{pq}+\frac{\sigma_q^2q^2}{2}V_{qq}+rpV_p+rqV_q-rV=0 \tag{5}$$

on the domain $(p,q,t)\in\left(0,\infty\right)^2\times[0,T)$.

The equivalent proof for an option on one asset using the replication method can be found in [**7**]. For the general case of two variables one can have a look at the martingale approach in [**3**], and for the PDEs one in [**12**]. The next result is a consequence of the existence and uniqueness of solutions for the initial value problem for second order parabolic PDEs. This result can be found, for instance, in [**8**] or [**9**].

COROLLARY 3.1. *Under the assumptions of Theorem 3.1, there exists a unique solution $V\left(p,q,t\right)$ of (5) with initial condition $V\left(p,q,T\right)=\Lambda\left(p,q\right)$.*

A particular case of Theorem 3.1 is when the coefficients appearing in the equation (5) can be assumed to be constant.

COROLLARY 3.2 (Constant coefficients). *Under the assumptions of Theorem 3.1, let us suppose that $r,\sigma_{p,q}$ are constant. Then the solution to*

$$\begin{cases} V_t+\frac{\sigma_p^2p^2}{2}V_{pp}+\rho\sigma_p\sigma_q pqV_{pq}+\frac{\sigma_q^2q^2}{2}V_{qq}+rpV_p+rqV_q-rV=0,\\ V(p,q,T)=\Lambda\left(p,q\right)\end{cases} \tag{6}$$

is given by

$$V\left(p,q,t\right)=\frac{p^\alpha q^\beta\exp\left(\gamma\sigma^2(T-t)/2\right)}{2\pi\sigma_p\sigma_q(T-t)\sqrt{1-\rho^2}}\int_0^{+\infty}\int_0^{+\infty}\Lambda\left(R,S\right)F\left(p,q;R,S\right)\mathrm{d}R\mathrm{d}S, \tag{7}$$

where

$$F\left(p,q;R,S\right)=R^{-\alpha-1}S^{-\beta-1}\exp\left(-\frac{\sigma_q^2\ln^2\left(\frac{p}{R}\right)-2\rho\sigma_p\sigma_q\ln\left(\frac{p}{R}\right)\ln\left(\frac{q}{S}\right)+\sigma_p^2\ln^2\left(\frac{q}{S}\right)}{2(T-t)\sigma_p^2\sigma_q^2\left(1-\rho^2\right)}\right),$$

with $\sigma^2=\sigma_p^2-2\rho\sigma_p\sigma_q+\sigma_q^2$ *and we have defined the parameters*

$$\begin{cases}\alpha=\dfrac{2\rho r\sigma_p+\sigma_p^2\sigma_q-\rho\sigma_p\sigma_q^2-2r\sigma_q}{2\sigma_q\sigma_p^2\left(1-\rho^2\right)},\\[2ex] \beta=\dfrac{2\rho r\sigma_q+\sigma_q^2\sigma_p-\rho\sigma_q\sigma_p^2-2r\sigma_p}{2\sigma_p\sigma_q^2\left(1-\rho^2\right)},\\[2ex] \gamma=-\dfrac{4\rho r\sigma_p\sigma_q+\sigma_p^2\sigma_q^2+4r^2}{4\left(1-\rho^2\right)\sigma_p^2\sigma_q^2}.\end{cases}\tag{8}$$

Proof. Let us assume that the exercise price is $E>0$ and let us define

$$\sigma^2=\sigma_p^2-2\rho\sigma_p\sigma_q+\sigma_q^2.$$

Notice that it is well-defined and non-negative. Now consider the change of variables

$$p=E\exp\left(x\right),\quad q=E\exp\left(y\right),\quad t=T-\frac{2}{\sigma^2}z,\quad V\left(p,q,t\right)=E\exp\left(\alpha x+\beta y+\gamma z\right)u\left(x,y,z\right).$$

With $V\left(p,q,T\right)=\Lambda\left(p,q\right)=u_0\left(x,y\right)=u\left(x,y,0\right)$. Notie that the new variables are dimensionless. In these variables we have that

$$\frac{\partial V}{\partial t}=-\frac{\sigma^2}{2}\frac{\partial V}{\partial z}=-\frac{\sigma^2}{2}E\exp\left(\alpha x+\beta y+\gamma z\right)\left(\gamma u+\frac{\partial u}{\partial z}\right),$$

$$\frac{\partial V}{\partial p}=\frac{1}{p}\frac{\partial V}{\partial x}=E\exp\left(\alpha x+\beta y+\gamma z\right)\frac{1}{p}\left(\alpha u+\frac{\partial u}{\partial x}\right),$$

$$\frac{\partial V}{\partial q}=\frac{1}{q}\frac{\partial V}{\partial y}=E\exp\left(\alpha x+\beta y+\gamma z\right)\frac{1}{q}\left(\beta u+\frac{\partial u}{\partial y}\right),$$

$$\frac{\partial^2 V}{\partial p^2}=E\exp\left(\alpha x+\beta y+\gamma z\right)\frac{1}{p^2}\left(\left(\alpha^2-\alpha\right)u+\left(2\alpha-1\right)\frac{\partial u}{\partial x}+\frac{\partial^2 u}{\partial x^2}\right),$$

$$\frac{\partial^2 V}{\partial q^2}=E\exp\left(\alpha x+\beta y+\gamma z\right)\frac{1}{q^2}\left(\left(\beta^2-\beta\right)u+\left(2\beta-1\right)\frac{\partial u}{\partial y}+\frac{\partial^2 u}{\partial y^2}\right),$$

$$\frac{\partial^2 V}{\partial p\partial q}=E\exp\left(\alpha x+\beta y+\gamma z\right)\frac{1}{pq}\left(\alpha\beta u+\alpha\frac{\partial u}{\partial y}+\beta\frac{\partial u}{\partial x}+\frac{\partial^2 u}{\partial x\partial y}\right).$$

Substituting in (5) and taking α,β,γ as in (8) we have that the terms below second order on x,y vanish, getting

$$u_z = \frac{\sigma_p^2}{\sigma^2} u_{xx} + 2\rho \frac{\sigma_p \sigma_q}{\sigma^2} u_{xy} + \frac{\sigma_q^2}{\sigma^2} u_{yy}. \tag{9}$$

Notice that the coefficients α, β are equivalents when exchanging p, q and that γ is symmetric in p, q. If we now do the further change of variables

$$x = \frac{\sigma_p}{\sigma} X, \quad y = \frac{\sigma_q}{\sigma} Y, \quad v(X, Y, z) = u(x, y, z),$$

with $v_0(X, Y) = u_0(x, y)$, the equation yields

$$v_z = v_{XX} + 2\rho v_{XY} + v_{YY}. \tag{10}$$

In order to classify the equation, we calculate $\Delta = AC - B^2$, with $A = 1, B = \rho, C = 1$. Thus,

$$\Delta = \left(1 - \rho^2\right)$$

is strictly positive, since we are assuming $-1 < \rho < 1$. We therefore conclude that we have a second order parabolic PDE on the variables (X, Y, z). If we consider the function

$$w(X, Y, z) = \frac{1}{4\pi\sqrt{1-\rho^2} z} \exp\left(-\frac{X^2 - 2\rho XY + Y^2}{4z\left(1-\rho^2\right)}\right),$$

it is immediate to see that w fulfills (10), and in particular

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} w(X, Y, z) \, \mathrm{d}X \mathrm{d}Y = 1$$

for all $z > 0$. Thus, we can recover any solutions to (9) such that $v(X, Y, 0) = v_0(X, Y)$ by convolving with w, as long as $v_0(X, Y)$ is regular enough (see [**8**]). Let us therefore write

$$v(X, Y, z) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} v_0(\chi, \eta) \, w(\chi - X, \eta - Y, z) \, \mathrm{d}\chi \mathrm{d}\eta.$$

Undoing the changes of variables and introducing integrating variables R, S we get (7).

□

Now we will proceed to derive a generalization of the former results. When a company creates a profit, it usually shares part of it with its shareholders. This is called dividend, and if the option is on assets that pay dividends, the risk-free construction needs to be adapted to the new type of assets. Thus, we will suppose that P, Q have dividends D_p, D_q which increase continuously, and the equivalent proposition to Proposition 3.1 is:

PROPOSITION 3.2. *Let us consider the Itô process given by*

$$\begin{cases} \mathrm{d}P = P\left(\left(\mu - D_p\right) \mathrm{d}t + \sigma_p \mathrm{d}X\right), \\ \mathrm{d}Q = Q\left(\left(\nu - D_q\right) \mathrm{d}t + \sigma_q\left(\rho \mathrm{d}X + \sqrt{1-\rho^2} \mathrm{d}Y\right)\right), \end{cases} \tag{11}$$

with $\sigma_{p,q}, D_{p,q}, \mu, \nu$ are deterministic functions of t and $-1 < \rho < 1$ constant, being X, Y independent Brownian motions. Then the solutions is given by

$$
\begin{aligned}
P =& P_0 \exp\left(\int_0^t \left(\mu(\tau) - D_p(\tau) - \sigma_p^2(\tau)/2\right) \mathrm{d}\tau + \int_0^t \sigma_p(\tau) \mathrm{d}X(\tau)\right), \\
Q =& Q_0 \exp\left(\int_0^t \left(\nu(\tau) - D_q(\tau) - \sigma_q^2(\tau)/2\right) \mathrm{d}\tau + \rho \int_0^t \sigma_q(\tau) \mathrm{d}X(\tau) \right.\\
&\left. + \sqrt{1-\rho^2} \int_0^t \sigma_q(\tau) \mathrm{d}Y(\tau)\right).
\end{aligned}
$$

PROOF. It follows from Proposition 3.1 taking $\mu' = \mu - D_p$ and $\nu' = \nu - D_q$. □

THEOREM 3.2. *Let us assume that the value of two assets follows the Itô process given in (11). Suppose that the market admits no arbitrages and that the value of an European option is a differentiable function $V(p,q,t)$. Let us further assume that the currency has a deterministic interest rate $r(t)$. Then V satisfies*

$$
V_t + \frac{\sigma_p^2 p^2}{2} V_{pp} + \rho \sigma_p \sigma_q p q V_{pq} + \frac{\sigma_q^2 q^2}{2} V_{qq} + (r - D_p) p V_p + (r - D_q) q V_q - rV = 0 \tag{12}
$$

on the domain $(p,q,t) \in (0,\infty)^2 \times [0,T)$.

The proof for the previous theorem can be found in [**12**] or [**13**]. Al results of existence and uniqueness of solutions for all values of the domain $(p,q,t) \in (0,\infty)^2 \times [0,T)$ can be adapted for this case. Next we will define the most usual kind of pay-offs for European options.

DEFINITION 3.3. The name for an European option presenting the pay-offs described by the following functions are named as follows

$$
\Lambda(p,q) = \begin{cases} \max\left(\max(p,q) - E, 0\right), \text{ maximum call,} \\ \max\left(E - \max(p,q), 0\right), \text{ maximum put,} \\ \max\left(\min(p,q) - E, 0\right), \text{ minimum call,} \\ \max\left(E - \min(p,q), 0\right), \text{ minimum put.} \end{cases}
$$

2. Exchange options. Similarity reduction

In this section we will study one kind of options that are of our interest since they are on two assets. Practically, they will be studied as if they were European options, since the difference they have with respect to the former ones does not appear in the deduction of the equation they have to fulfill, but it is in fact a characterization on their pay-off.

DEFINITION 3.4. An exchange option is an option that gives the holder the right to exchange an asset for another one at the expiry date.

LEMMA 3.1. *An exchange option has final condition* $\Lambda(p,q) = \max(p-q, 0)$.

PROOF. Of course, the holder will take the asset P if the value $P > Q$ obtaining a profit $P - Q$ and otherwise will not. □

The next argument can be generalized when the final conditions Λ are homogeneous of a given degree. In this case, since our condition is homogeneous of degree 1, we look for solutions of the form

$$V(p,q,t) = qH(s,t),$$

where

$$s = \frac{p}{q}.$$

Thus, knowing that the deduction of (12) we have not used anything related to the pay-off, except that we have required it is non-vanishing on a set big enough, an exchange option needs to fulfill equation (12) as well.

PROPOSITION 3.3. *Let us suppose that the value of two assets follows (11). Let us write $s = p/q$. Let us suppose that $V(p,q,t) = qH(s,t)$ with $\Lambda(p,q) = \max(p-q,0)$. Then $H(s,t)$ fulfills*

$$H_t + \frac{\sigma^2 s^2}{2} H_{ss} + (D_q - D_p) s H_s - D_q H = 0 \tag{13}$$

with final condition $H(s,T) = \max(s-1,0)$. Where we have defined

$$\sigma^2 = \sigma_p^2 - 2\rho\sigma_p\sigma_q + \sigma_q^2.$$

PROOF. Let us consider equation (12) and that $V(p,q,t) = qH(s,t)$. Let us investigate the change of variables

$$\frac{\partial qH}{\partial p} = q\frac{\partial s}{\partial p}\frac{\partial H}{\partial s} = \frac{\partial H}{\partial s}, \qquad \frac{\partial^2 qH}{\partial p^2} = \frac{1}{q}\frac{\partial^2 H}{\partial s^2},$$

$$\frac{\partial qH}{\partial q} = H + q\frac{\partial s}{\partial q}\frac{\partial H}{\partial s} = H - \frac{p}{q}\frac{\partial H}{\partial s} = H - s\frac{\partial H}{\partial s},$$

$$\frac{\partial^2 qH}{\partial q^2} = -\frac{s}{q}\frac{\partial H}{\partial s} - \left(-\frac{s}{q}\frac{\partial H}{\partial s} - \frac{s^2}{q}\frac{\partial^2 H}{\partial s^2}\right) = \frac{s^2}{q}\frac{\partial^2 H}{\partial s^2},$$

$$\frac{\partial^2 qH}{\partial q \partial p} = -\frac{s}{q}\frac{\partial^2 H}{\partial s^2}, \qquad \frac{\partial qH}{\partial t} = q\frac{\partial H}{\partial t}.$$

Substituting in equation (12) and dividing by q, after simplifying we obtain that

$$H_t + \left(\sigma_p^2 - 2\rho\sigma_p\sigma_q + \sigma_q^2\right)\frac{s^2}{2}H_{ss} + s(D_q - D_p)H_s - D_q H = 0.$$

We can define

$$\sigma^2 = \sigma_p^2 - 2\rho\sigma_p\sigma_q + \sigma_q^2.$$

And obtain that H fulfills (13). Moreover, since $V(p,q,T) = \max(p-q,0) = qH(s,T)$, dividing by q we have that

$$H(s,T) = \max(s-1,0).$$

□

Note that an exchange option can be determined by using the Black-Scholes of one variable with volatility σ, interest rate D_q and dividend D_p. The later proof can be found in [**3**] or [**14**].

A particular case and rather known of the former one is the Margrabe's formula, which is the solution to the equation assuming constant coefficients and no dividends.

COROLLARY 3.3 (Margrabe's formula). *Let us assume that we are under the conditions of Proposition 3.3 and the coeficients $\sigma_{p,q}$ are constant and $D_{p,q} = 0$. Then V is given by*

$$V(p,q,t) = pN(d_1) - qN(d_2), \tag{14}$$

where

$$d_1 = \frac{\ln(p/q) + \sigma^2 (T-t)/2}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\ln(p/q) - \sigma^2 (T-t)/2}{\sigma\sqrt{T-t}},$$

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x.$$

The proof is based on the resolution of an European option with one asset and pay-off $\max(s-1, 0)$. For the explicit proof one can check [**10**], [**13**] or [**14**].

3. Barrier options

We now introduce a few concepts and definitions that will be used later on.

DEFINITION 3.5. A barrier option on two assets is an option with the same characteristics that the European but with the condition that if the assets are below or above a given value, the value of the option is zero. This given value is called barrier.

LEMMA 3.2. *Let us assume we have an option with barrier on two assets P, Q. Let us assume that the barriers are on $P = a$ and $Q = b$. Then the option needs to satisfy the boundary conditions*

$$\begin{cases} V(a,q,t) = 0, \\ V(p,b,t) = 0. \end{cases} \tag{15}$$

PROOF. Of course, if the barrier is on $P = a$ i $Q = b$, the value of the option needs to be zero, since the value on one of the two sides of the barrier is always zero and the value of the option needs to be continuous. □

Thus, if we want to solve the problem (12), we would have to add the conditions (15).

4. Perpetual options

In this section we will study a type of options that is slightly different from the former ones. That is because the problem we will end up posing in order to find the value of the option will bring us to a free-boundary problem, usually found when studying American options.

Definition 3.6. A perpetual option is an option for which the expiry date and strike price are not fixed.

A result that characterizes this kind of derivatives is the following one:

Theorem 3.3. *The value of a perpetual option, V, on two assets as the ones described in (11) and pay-off $\Lambda(p,q)$, fulfills the non-linear problem*

(16)
$$\left(\frac{\sigma_p^2}{2}V_{pp}+\rho\sigma_p\sigma_q V_{pq}+\frac{\sigma_q^2}{2}V_{qq}+(r-D_p)V_p+(r-D_q)V_q-rV\right)(\Lambda-V)=0,$$

with

$$\frac{\sigma_p^2}{2}V_{pp}+\rho\sigma_p\sigma_q V_{pq}+\frac{\sigma_q^2}{2}V_{qq}+(r-D_p)V_p+(r-D_q)V_q-rV\leq 0,$$

$$\Lambda-V\leq 0.$$

Moreover, the contact between Λ and V needs to be of class $\mathcal{C}^1$.

The proof of the theorem above is not the aim of the project and we will not give it. A proof for a concreet case of a maximum put perpetual option can be found in [**15**]. A more profound discussion of the topic that encloses our case and deals with American options with more than one asset can be found in [**16**].

CHAPTER 4

Analysis of the cases $P, Q = 0$ and $\rho^2 = 1$

We will now investigate how the equation behaves in three particular cases. Through the project we have assumed that we are working on two positive assets, but what would happen if one of them is zero? What we would like is that when P or Q go to zero, we recover the equation with only one variable (the non-vanishing one), i.e. there is a continuity on P and Q so that we obtain the original Black-Scholes equation. Another case to consider is what happens when $\rho^2 = 1$. Is there a continuity on the equation when the assets exhibit a total correlation? That is, if $\rho^2 = 1$, the answer is so simple as to put into the equation $\rho = \pm 1$? Next we will answer these questions.

1. Case $P, Q = 0$

We can go to the general case of the assets described in (11). If $P = 0$, the stochastic differential equation becomes

$$\begin{cases} dP = 0, \\ dQ = Q\left(\left(\nu - D_q\right)\mathrm{d}t + \sigma_q \mathrm{d}W\right), \end{cases} \tag{17}$$

where $\mathrm{d}W = \rho \mathrm{d}X + \sqrt{1-\rho^2}\mathrm{d}Y$. Thus, we need to see whether this fact will affect the proof of Theorem 3.2. In fact, the construction of the risk-free portfolio with a zero asset does not change the procedure and we end up proving the Black-Scholes equation for one asset.

Proposition 4.1. *Let us assume that the value of two assets P, Q is described as in (17) with $P = 0$ and that the market admits no arbitrages. Let us further assume that the value of an option is a differentiable function $V(p, q, t)$ and that the money in a bank earns money at deterministic interest rate $r(t)$. Then V fulfills*

$$V_t + \frac{\sigma_q^2 q^2}{2} V_{qq} + \left(r - D_q\right) q V_q - rV = 0. \tag{18}$$

Moreover $\Lambda(p, q) = \Lambda(0, q) = \lambda(q)$.

Thus, we can conclude that the equation is continuous with respect to p, since if we put $p = 0$ in (12) we obtain (18). Similarly we obtain the analogous result for q. If we assume that $Q = 0$ then we have

$$\begin{cases} dP = P\left(\left(\mu - D_p\right)\mathrm{d}t + \sigma_q \mathrm{d}X\right), \\ dQ = 0, \end{cases} \tag{19}$$

And therefore

PROPOSITION 4.2. *Let us assume that the value of two assets P, Q is described as in (19) with $Q = 0$ and that the market admits no arbitrages. Let us further assume that the value of an option is a differentiable function $V(p, q, t)$ and that the money in a bank earns money at deterministic interest rate $r(t)$. Then V fulfills*

$$V_t + \frac{\sigma_p^2 p^2}{2} V_{pp} + (r - D_p)\, p V_p - rV = 0. \tag{20}$$

Moreover $\Lambda(p, q) = \Lambda(p, 0) = \lambda(p)$.

Even though that *a priori* it may seem that discussing this is highly theoretical and has not much interest, it will be important when solving the equations numerically. The proofs of both propositions are, essentially, the proof of the Black-Scholes equation for one asset and it can be found in [**3**], [**7**] or in [**10**].

2. Case $\rho^2 = 1$

Now we will study the pathological case in which the two Brownian motions have a correlation coefficient such that $\rho^2 = 1$. If this is the case, let us consider the Itô process given by

$$\begin{cases} \mathrm{d}P = P\left((\mu - D_p)\,\mathrm{d}t + \sigma_p \mathrm{d}X\right), \\ \mathrm{d}Q = Q\left((\nu - D_q)\,\mathrm{d}t + \sigma_q \rho \mathrm{d}X\right), \end{cases} \tag{21}$$

with $\sigma_{p,q}, D_{p,q}, \mu, \nu$ deterministic functions of t. These two processes are (11) assuming $\rho^2 = 1$, thus, $\rho = \pm 1$. In order to have an easy-going procedure, we will suppose that $\sigma_p \neq \sigma_q$ and that $P_0, Q_0 \neq 0$. In order to make calculations faster, it is important bearing in mind that in our case $\rho = 1/\rho$.

Let us consider the variable

$$\frac{\mathrm{d}M}{M} = \frac{\sigma_p - \rho\sigma_q}{2}\left(\frac{1}{\sigma_p}\frac{\mathrm{d}P}{P} - \rho\frac{1}{\sigma_q}\frac{\mathrm{d}Q}{Q}\right). \tag{22}$$

Calculating, we obtain that the process we obtain is deterministic

$$\frac{\mathrm{d}M}{M} = f(t)\mathrm{d}t, \tag{23}$$

where

$$f(t) = \frac{\sigma_p - \rho\sigma_q}{2}\left(\frac{\mu - D_p}{\sigma_p} - \rho\frac{\nu - D_q}{\sigma_q}\right).$$

Similarly, we can construct a new Itô process defined by

$$\frac{\mathrm{d}S}{S} = \frac{\sigma_p + \rho\sigma_q}{2}\left(\frac{1}{\sigma_p}\frac{\mathrm{d}P}{P} + \rho\frac{1}{\sigma_q}\frac{\mathrm{d}Q}{Q}\right), \tag{24}$$

so that

$$\frac{\mathrm{d}S}{S} = a(t)\mathrm{d}t + b(t)\mathrm{d}X, \tag{25}$$

where

$$a(t) = \frac{\sigma_p + \rho\sigma_q}{2}\left(\frac{\mu - D_p}{\sigma_p} + \rho\frac{\nu - D_q}{\sigma_q}\right), \quad b(t) = \sigma_p + \rho\sigma_q.$$

Thus, we can consider that the value of the option depends on these two new variables, that is, $V(S, M, t)$. Let us now investigate the equation the needs to be fulfilled.

THEOREM 4.1. *Suppose that M and S are defined as in (23) and (25) respectively and that the market admits no arbitrages. Let us assume that the value of an European option is a differentiable function $V(s, m, t)$ and that $V(s, m, T) = \Lambda(s, m)$. Then V fulfills the equation*

$$V_t + fmV_m + \frac{b^2 s^2}{2}V_{ss} + rsV_s - rV = 0, \tag{26}$$

where, as before,

$$f(t) = \frac{\sigma_p - \rho\sigma_q}{2}\left(\frac{\mu - D_p}{\sigma_p} - \rho\frac{\nu - D_q}{\sigma_q}\right),$$
$$b(t) = \sigma_p + \rho\sigma_q.$$

PROOF. Let us consider a hedging portfolio $\Pi = \Gamma V - \Delta S$, so that by being self-financing $\mathrm{d}\Pi = \Gamma \mathrm{d}V - \Delta \mathrm{d}S$. If we apply (3) to $V(s, m, t)$ and evaluate $\mathrm{d}V, \mathrm{d}S$ in $\mathrm{d}\Pi$ we have

$$\mathrm{d}\Pi = \Gamma\left(V_t + m_t V_m + \frac{b^2 s^2}{2}V_{ss}\right)\mathrm{d}t + aS\left(\Gamma V_s - \Delta\right)\mathrm{d}t + bS\left(\Gamma V_s - \Delta\right)\mathrm{d}X.$$

By imposing that it is risk-free, the stochastic part needs to vanish, so

$$\Delta = \Gamma V_s.$$

Thus, by Proposition 2.3, $\mathrm{d}\Pi = r\Pi\mathrm{d}t$, obtaining V needs to fulfill

$$V_t + m_t V_m + \frac{b^2 s^2}{2}V_{ss} + rsV_s - rV = 0.$$

If we now use that $m_t = mf(t)$ and we evaluate that into the equation above, we obtain the desired result

$$V_t + fmV_m + \frac{b^2 s^2}{2}V_{ss} + rsV_s - rV = 0.$$

□

The reason why we did not use the same methodology as it is used to prove Theorem 3.1 or 3.2 is because when constructing the hedging portfolio $\Pi = cV - aP - bQ$ and imposing that it is risk-free, we have no information enough as to establish a relation between the holding of asset a, b and option c. This extra degree of freedom makes the argument fail. This is why we derive these two new elements, M and S, a deterministic one and another stochastic. By doing that, we can construct a hedging portfolio Π as we have done in the proof, which only depends on the asset and the option and we are able to eliminate the stochastic component when imposing that it is risk-free. We can therefore conclude that equation (12) is not continuous in the case $\rho \to \pm 1$.

PROPOSITION 4.3. *Given the assets P, Q described in (21), let us switch to the variables M, S given in (22) and (24) respectively. Let us further assume that $\sigma_p, \sigma_q, \mu, \nu$ are constant and that the assets pay no dividends, $D_{p,q} = 0$. Then we have*

$$\begin{cases} M\left(P,Q,t\right) = M_0 \left(P^{1/\sigma_p} Q^{-\rho/\sigma_q}\right)^{\sigma_-/2} \exp\left(\dfrac{\sigma_-^2}{4} t\right), \\ \\ S\left(P,Q,t\right) = S_0 \left(P^{1/\sigma_p} Q^{\rho/\sigma_q}\right)^{\sigma_+/2} \exp\left(\dfrac{-\sigma_+^2}{4} t\right), \end{cases} \tag{27}$$

where $\sigma_\pm = \sigma_p \pm \rho\sigma_q$.

PROOF. Let us write

$$\sigma_\pm = \sigma_p \pm \rho\sigma_q$$

From (22) and (24) we have

$$\begin{aligned} \frac{\mathrm{d}M}{M} &= \frac{\sigma_-}{2}\left(\frac{1}{\sigma_p}\frac{\mathrm{d}P}{P} - \rho\frac{1}{\sigma_q}\frac{\mathrm{d}Q}{Q}\right), \\ \frac{\mathrm{d}S}{S} &= \frac{\sigma_+}{2}\left(\frac{1}{\sigma_p}\frac{\mathrm{d}P}{P} + \rho\frac{1}{\sigma_q}\frac{\mathrm{d}Q}{Q}\right) \end{aligned}$$

and from (23) and (25)

$$\begin{aligned} \frac{\mathrm{d}M}{M} &= f(t)\mathrm{d}t, \\ \frac{\mathrm{d}S}{S} &= a(t)\mathrm{d}t + b(t)\mathrm{d}X, \end{aligned}$$

where a, b are

$$\begin{aligned} a &= \frac{\sigma_+}{2}\left(\frac{\mu}{\sigma_p} + \rho\frac{\nu}{\sigma_q}\right), \\ b &= \sigma_+. \end{aligned}$$

To integrate the processes M, S we need to use (3). Since M is deterministic, $\mathrm{d}M\mathrm{d}M = 0$ whilst $\mathrm{d}S\mathrm{d}S = S^2\sigma_+^2\mathrm{d}t$. Moreover, $\mathrm{d}P\mathrm{d}P = P^2\sigma_p^2\mathrm{d}t$ and $\mathrm{d}Q\mathrm{d}Q = Q^2\sigma_q^2\mathrm{d}t$. Then,

$$\begin{aligned} \mathrm{d}\left(\ln\left(P\right)\right) &= \frac{\mathrm{d}P}{P} - \frac{\sigma_p^2}{2}\mathrm{d}t, & \mathrm{d}\left(\ln\left(Q\right)\right) &= \frac{\mathrm{d}Q}{Q} - \frac{\sigma_q^2}{2}\mathrm{d}t, \\ \mathrm{d}\left(\ln\left(M\right)\right) &= \frac{\mathrm{d}M}{M}, & \mathrm{d}\left(\ln\left(S\right)\right) &= \frac{\mathrm{d}S}{S} - \frac{\sigma_+^2}{2}\mathrm{d}t. \end{aligned}$$

Thus, we add and rest wherever there is a stochastic process its quadratic variation, so that

$$\frac{\mathrm{d}M}{M} = \frac{\sigma_-}{2}\left(\frac{1}{\sigma_p}\left(\frac{\mathrm{d}P}{P} - \frac{\sigma_p^2}{2}\mathrm{d}t\right) + \frac{\sigma_p}{2}\mathrm{d}t - \frac{\rho}{\sigma_q}\left(\frac{\mathrm{d}Q}{Q} - \frac{\sigma_q^2}{2}\mathrm{d}t\right) - \frac{\rho\sigma_q}{2}\mathrm{d}t\right)$$

And the equation can be trivially integrated obtaining

$$\ln(M/M_0) = \frac{\sigma_-}{2}\ln\left(P^{1/\sigma_p}Q^{-\rho/\sigma_q}\right) + \frac{\sigma_-^2}{4}t.$$

That is,

$$M\left(P,Q,t\right) = M_0(P^{1/\sigma_p}Q^{-\rho/\sigma_q})^{\sigma_-/2}\exp\left(\frac{\sigma_-^2}{4}t\right).$$

Similarly with S we have

$$\frac{\mathrm{d}S}{S} - \frac{\sigma_+^2}{2}\mathrm{d}t + \frac{\sigma_+^2}{2}\mathrm{d}t = \\ \frac{\sigma_+}{2}\left(\frac{1}{\sigma_p}\left(\frac{\mathrm{d}P}{P} - \frac{\sigma_p^2}{2}\mathrm{d}t\right) + \frac{\sigma_p}{2}\mathrm{d}t + \rho\frac{1}{\sigma_q}\left(\frac{\mathrm{d}Q}{Q} - \frac{\sigma_q^2}{2}\mathrm{d}t\right) + \frac{\rho\sigma_q}{2}\mathrm{d}t\right).$$

It can be also trivially integrated

$$\ln(S/S_0) = \frac{\sigma_+}{2}\ln\left(P^{1/\sigma_p}Q^{\rho/\sigma_q}\right) - \frac{\sigma_+^2}{4}t.$$

That is,

$$S\left(P,Q,t\right) = S_0(P^{1/\sigma_p}Q^{\rho/\sigma_q})^{\sigma_+/2}\exp\left(-\frac{\sigma_+^2}{4}t\right).$$

□

We now give a result that provides a way of fixing M_0, S_0.

LEMMA 4.1. *Consider equation (26), and the change $x = \alpha m$ and $y = \beta s$ with α, β non-zero constant. Let us denote by $W\left(x,y,t\right) = V\left(m,s,t\right)$. Then W fulfills again (26) in the variables x, y, t.*

PROOF. Consider the said change of variables and the chain rule

$$\frac{\partial}{\partial m} = \alpha\frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial s} = \beta\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial m} = \beta^2\frac{\partial}{\partial y^2}.$$

Evaluate in equation (26) the change

$$W_t + f\frac{x}{\alpha}\alpha W_x + \frac{b^2y^2}{2\beta^2}\beta^2 W_{yy} + r\frac{y}{\beta}\beta W_y - rW = 0.$$

Canceling terms,

$$W_t + fxW_x + \frac{b^2y^2}{2}W_{yy} + ryW_y - rW = 0.$$

□

We conclude that a re-scaling of the variables s, m does not affect the equation. Thus, we can take $M_0 = 1, S_0 = 1$ from now on. Now we can consider how it would be, for instance, the problem of a maximum European put option. Let us consider the change of variables given by (27) and find P, Q as functions of M, S, so that

$$P = \left(S^{1/\sigma_+} M^{1/\sigma_-}\right)^{\sigma_p} \exp\left(\frac{\rho\sigma_p\sigma_q}{2} t\right),$$

$$Q = \left(S^{1/\sigma_+} M^{-1/\sigma_-}\right)^{\rho\sigma_q} \exp\left(\frac{\rho\sigma_p\sigma_q}{2} t\right).$$

Then, we can take the change of variables induced by

$$\varphi(s, m, t) = (p(s, m, t), q(s, m, t), t),$$

for the set of points such that $t = T$, we have

$$\varphi(s, m, T) = (p(s, m, T), q(s, m, T), T).$$

Thus, we can change the value of the pay-off for a maximum put option

$$\Lambda(s, m) := V(p(s, m, T), q(s, m, T), T) = \max(E - \max(p(s, m, T), q(s, m, T)), 0) =$$

$$\max\left(E - \max\left(\left(s^{1/\sigma_+} m^{1/\sigma_-}\right)^{\sigma_p} \exp\left(\frac{\rho\sigma_p\sigma_q T}{2}\right), \left(s^{1/\sigma_+} m^{-1/\sigma_-}\right)^{\rho\sigma_q} \exp\left(\frac{\rho\sigma_p\sigma_q T}{2}\right)\right), 0\right).$$

If we want to solve the problem assuming constant coefficients and that the assets pay no dividends with $\rho^2 = 1$, we should solve the final value problem

$$\left\{ \begin{array}{l} V_t + fmV_m + \dfrac{b^2 s^2}{2} V_{ss} + rsV_s - rV = 0, \\ V(s, m, T) = \Lambda(s, m). \end{array} \right. \tag{28}$$

CHAPTER 5

Resolution by using numerical methods

In this chapter we will solve an European option and a perpetual option by using numerical methods. To do so, we will have to give a pay-off as one defined in Definition 3.3. Moreover, we will also have to impose boundary conditions, since in order to integrate numerically the PDE we need to do it in a bounded domain.

1. European option

Let us begin by introducing the method of the explicit finite differences in the case of the European option. In order to make calculations easier, we will assume that the assets pay no dividends. In a first approach to the problem, we will deal with European options with pay-off $\Lambda(p,q)$. Assuming that, let us determine the boundary conditions for the option. Let us first consider the values close to zero. If the asset value is arbitrarily small, we need to see what happens with

$$\lim_{p\to 0} V(p,q,t) \text{ i } \lim_{q\to 0} V(p,q,t).$$

In the Propositions 4.1 and 4.2 we have seen that in the case that one of the assets is zero, then the function satisfies the Black-Scholes equation for the non-vanishing asset, so that

$$V(0,q,t) = VQ(q,t)\,, \quad V(p,0,t) = VP(p,t)\,,$$

where VP, VQ are the solutions to the Black-Scholes equation with one asset for the cases $q=0, p=0$ respectively. Furthermore, the pay-off of this option will be the one of V in p or q zero.

We need also to see how the other side of the domain behaves. If the value of the assets is arbitrarily large, we can consider that what happens with the other one is not relevant and, thus, it will only matter the large one. In other words,

$$V(p,q,t) \sim VP(p,t) \text{ if } p\to +\infty, \quad V(p,q,t) \sim VQ(q,t) \text{ if } q\to +\infty,$$

with VP, VQ defined as before. Notice that in order to solve the problem in two variables we first need to solve it with one so we can give the boundary conditions.

We now have all we need to solve numerically the problem. We will assume that the coefficients involved are constant and, as in the proof of Corollary 8, we can reduce the problem to equation (9), given by

$$u_z = \frac{\sigma_p^2}{\sigma^2}u_{xx} + 2\rho\frac{\sigma_p\sigma_q}{\sigma^2}u_{xy} + \frac{\sigma_q^2}{\sigma^2}u_{yy}.$$

We will solve the problem in the domain $(x,y,z) \in [a,c] \times [b,d] \times [0,Z]$, where the constants a,b,c,d,Z are given by the change of variables in terms of the assets p,q and the time t. Let us consider a regular partition of the aforementioned intervals, so that

$$R = \{(x_i, y_j, z_k) \ : \ x_i = i\mathrm{d}x, \ i = -L_m \div L_p; \ y_j = j\mathrm{d}y, \\ j = -M_m \div M_p; \ z_k = k\mathrm{d}z, \ k = 0 \div N\},$$

where L_m, L_p, M_m, M_p, N are natural numbers and $\mathrm{d}x, \mathrm{d}y, \mathrm{d}z$ satisfy, respectively, that

$$-L_m\mathrm{d}x = a, \ L_p\mathrm{d}x = c, \ -M_m\mathrm{d}y = b, \ M_p\mathrm{d}y = d, \ N\mathrm{d}z = Z.$$

Thus, we can discretize the equation on the mesh, so that

$$\frac{u_{i,j}^{k+1} - u_{i,j}^k}{\mathrm{d}z} = \frac{\sigma_p^2}{\sigma^2}\frac{u_{i+1,j}^k - 2v_{i,j}^k + u_{i-1,j}^k}{\mathrm{d}x^2} + \frac{\sigma_q^2}{\sigma^2}\frac{u_{i,j+1}^k - 2u_{i,j}^k + u_{i,j-1}^k}{\mathrm{d}y^2} + \\ + 2\rho\frac{\sigma_p\sigma_q}{\sigma^2}\frac{u_{i+1,j+1}^k - u_{i+1,j-1}^k - u_{i-1,j+1}^k + u_{i-1,j-1}^k}{4\mathrm{d}x\mathrm{d}y}.$$

We can express the former equation for the time $k+1$ in terms of the former times,

$$u_{i,j}^{k+1} = (1 - 2\varepsilon_x - 2\varepsilon_y)\, u_{i,j}^k + \varepsilon_x \left(u_{i+1,j}^k + u_{i-1,j}^k\right) + \varepsilon_y \left(u_{i,j+1}^k + u_{i,j-1}^k\right) + \\ + \frac{\rho\varepsilon_{xy}}{2}\left(u_{i+1,j+1}^k - u_{i+1,j-1}^k - u_{i-1,j+1}^k + u_{i-1,j-1}^k\right),$$

where, naturally, we have defined

$$\varepsilon_x = \frac{\sigma_p^2}{\sigma^2}\frac{\mathrm{d}z}{\mathrm{d}x^2}, \quad \varepsilon_y = \frac{\sigma_q^2}{\sigma^2}\frac{\mathrm{d}z}{\mathrm{d}y^2}, \quad \varepsilon_{xy} = \frac{\sigma_p\sigma_q}{\sigma^2}\frac{\mathrm{d}z}{\mathrm{d}x\mathrm{d}y}.$$

The convergence of this method is determined, in our case, by these quantities, that is, the method will converge if

$$\varepsilon_x + \varepsilon_y + \frac{\rho}{2}\varepsilon_{xy} < \frac{1}{2}$$

and it will diverge if the expression is greater than 0.5 (see [**11**]).

In order to solve the problem, we will start with $k = 0$, where we know the solution corresponding to the initial condition, where we need to calculate $u_{i,j}^0 = u_0(i\mathrm{d}x, j\mathrm{d}y)$ for all i, j. For the boundaries of the domain we have the conditions induced by the change of variables

$$u(a,y,z) = f_a(y,z), \quad u(c,y,z) = f_c(y,z),$$

$$u(x,b,z) = f_b(x,z), \quad u(x,d,z) = f_d(x,z).$$

Thus, we can find the values $u_{-L_m,j}^k, u_{L_p,j}^k$ and $u_{i,-M_m}^k, u_{i,M_p}^k$ for all k. Now we only need to iterate the method on the unknown values to determine the nodes. Undoing the change of variables on u we find V as a function of the initial and boundary conditions we have given.

Now we give an example. Consider the maximum put option $\mathcal{P}$ with pay-off

$$\Lambda\left(p,q\right)=\max\left(E-\max\left(p,q\right),0\right),$$

with expiry date $T=0.5\,years$, strike price $E=6$, volatilities $\sigma_p=0.2$ years$^{-1/2}$, $\sigma_q=0.13$ years$^{-1/2}$, correlation coefficient $\rho=0.35$ and interest rate $r=0.05$ years^{-1}. Establish the limits of the rectangle on the boundaries of P given by $\{A=2,C=8\}$ and the boundaries given by Q given by $\{B=2,D=8\}$. Choose $L_m=40, M_m=40, N=80$ and we have that $L_p=10, M_p=10$. Moreover, we have the boundary conditions

$$F_A\left(A,q,t\right)=\mathcal{P}Q\left(q,t\right),\quad F_B\left(p,B,t\right)=\mathcal{P}P\left(p,t\right),$$

$$F_C\left(C,q,t\right)=\mathcal{P}P\left(C,t\right),\quad F_D\left(p,D,t\right)=\mathcal{P}Q\left(D,t\right),$$

where $\mathcal{P}P,\mathcal{P}Q$ have initial condition

$$\lambda\left(G\right)=\max\left(E-G,0\right).$$

Now we show the graphs corresponding to two boundary conditions, Fig. 1 and 2.

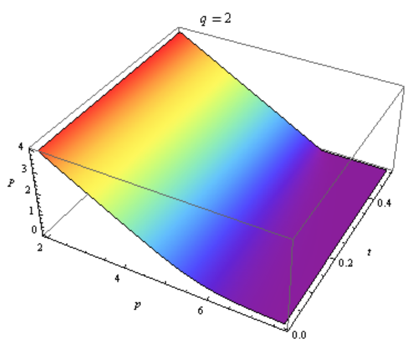

FIGURE 1. Surface corresponding to $\mathcal{P}P\left(p,t\right)$.

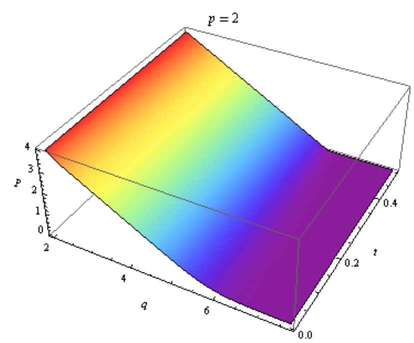

FIGURE 2. Surface corresponding to $\mathcal{P}Q\left(q,t\right)$.

Now we show the different values for the option in function of p,q for the fixed times $t=0$ and expiry date $t=0.5$, Fig. 3 and Fig. 4 respectively.

Let us finally compare the value of the numerical approximation with the numerical approximation of the formula (7), which is the integral solution of the equation,

$$\mathcal{P}_{exact}\left(3.974027,3.974027,0.25\right)=1.795541$$
$$\mathcal{P}_{numeric}\left(3.974027,3.974027,0.25\right)=1.795876$$

The result is very good since we have four significant figures.

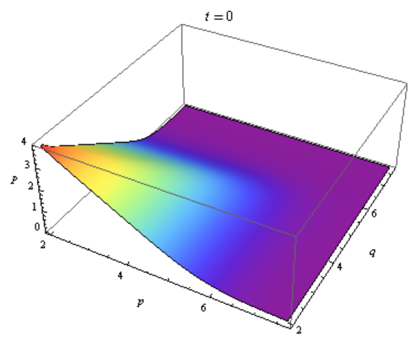

FIGURE 3. Surface corresponding to $\mathcal{P}(p,q,0)$.

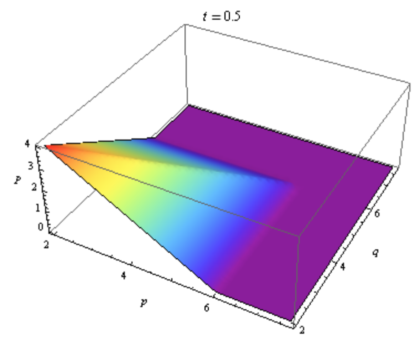

FIGURE 4. Surface corresponding to $\mathcal{P}(p,q,T)$.

2. Perpetual option

Now we will solve the problem for a maximum perpetual put option with boundaries on P and Q. By Theorem 3.3 we know that $\mathcal{P}$ needs to fulfill (16). By Lemma 3.2 we have to impose the boundary that on the boundary conditions (15) is fulfilled. We will assume that the coefficients are constant and the assets pay no dividends.

The problem to be solved is

$$\left(\frac{\sigma_p^2 p^2}{2}\mathcal{P}_{pp} + \rho\sigma_p\sigma_q pq\mathcal{P}_{pq} + \frac{\sigma_q^2 q^2}{2}\mathcal{P}_{qq} + rp\mathcal{P}_p + rq\mathcal{P}_q - r\mathcal{P}\right)(\Lambda - \mathcal{P}) = 0.$$

With condition

$$\Lambda(p,q) = \max\left(E - \max(p,q), 0\right)$$

and boundaries on $P = C, Q = D$. Thus, we first establish the boundary conditions for our problem. There are two that are trivial since they are being imposed by us,

$$\mathcal{P}(c,q) = 0, \qquad \mathcal{P}(p,d) = 0.$$

However, the other extremes, close to the origin, we will proceed with the same argument as before and we will take

$$\mathcal{P}(p,0) = \mathcal{P}P(p), \qquad \mathcal{P}(0,q) = \mathcal{P}Q(q),$$

where $\mathcal{P}P, \mathcal{P}Q$ are the solutions to the problem with one variable. Moreover, they have pay-off

$$\lambda(G) = \max(E - G).$$

Now we will go for an easier expression. Consider the change of variables

$$p = E\exp(x), \quad q = E\exp(y), \quad V(p,q) = E\exp(\alpha x + \beta y)\, u(x,y).$$

Using the chain rule,

$$\frac{\partial V}{\partial p} = \frac{1}{p}\frac{\partial V}{\partial x} = E\exp\left(\alpha x+\beta y\right)\frac{1}{p}\left(\alpha u+\frac{\partial u}{\partial x}\right),$$

$$\frac{\partial V}{\partial q} = \frac{1}{q}\frac{\partial V}{\partial y} = E\exp\left(\alpha x+\beta y\right)\frac{1}{q}\left(\beta u+\frac{\partial u}{\partial y}\right),$$

$$\frac{\partial^2 V}{\partial p^2} = E\exp\left(\alpha x+\beta y\right)\frac{1}{p^2}\left(\left(\alpha^2-\alpha\right)u+\left(2\alpha-1\right)\frac{\partial u}{\partial x}+\frac{\partial^2 u}{\partial x^2}\right),$$

$$\frac{\partial^2 V}{\partial q^2} = E\exp\left(\alpha x+\beta y\right)\frac{1}{q^2}\left(\left(\beta^2-\beta\right)u+\left(2\beta-1\right)\frac{\partial u}{\partial y}+\frac{\partial^2 u}{\partial y^2}\right),$$

$$\frac{\partial^2 V}{\partial p\partial q} = E\exp\left(\alpha x+\beta y\right)\frac{1}{pq}\left(\alpha\beta u+\alpha\frac{\partial u}{\partial y}+\beta\frac{\partial u}{\partial x}+\frac{\partial^2 u}{\partial x\partial y}\right).$$

Thus, if we take

$$\alpha = \frac{\sigma_q\sigma_p^2+2\rho\sigma_p r-\rho\sigma_p\sigma_q^2-2\sigma_q r}{2\sigma_p^2\sigma_q\left(1-\rho^2\right)},$$

$$\beta = \frac{\sigma_p\sigma_q^2+2\rho\sigma_q r-\rho\sigma_q\sigma_p^2-2\sigma_p r}{2\sigma_q^2\sigma_p\left(1-\rho^2\right)}$$

the problem becomes

$$\left(\frac{\sigma_p^2}{2}u_{xx}+\rho\sigma_p\sigma_q u_{xy}+\frac{\sigma_q^2}{2}u_{yy}-cu\right)\left(u_0-u\right)=0,$$

with $\sigma^2=\sigma_p^2+\sigma_q^2-2\rho\sigma_p\sigma_q$ and

$$c = \frac{\left(4r^2+\sigma_p^2\sigma_q^2+4\rho r\sigma_p\sigma_q\right)\sigma^2}{8\left(1-\rho^2\right)\sigma_p^2\sigma_q^2}.$$

Moreover, it needs to be fulfilled

$$\frac{\sigma_p^2}{2}u_{xx}+\rho\sigma_p\sigma_q u_{xy}+\frac{\sigma_q^2}{2}u_{yy}-cu\le 0,\quad u_0-u\le 0.$$

If we discretize te equation as before, we have

$$\left(\frac{\sigma_p^2}{2}\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{\mathrm{d}x^2}+\rho\sigma_p\sigma_q\frac{u_{i+1,j+1}-u_{i+1,j-1}-u_{i-1,j+1}+u_{i-1,j-1}}{4\mathrm{d}x\mathrm{d}y}\right.$$

$$\left.+\frac{\sigma_q^2}{2}\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{\mathrm{d}y^2}-cu_{i,j}\right)\left(\left(u_0\right)_{i,j}-u_{i,j}\right)=0.$$

We first calculate the boundary values $i=-L_m,L_p$ for all j and $j=-M_m,M_p$ for all i. We will aslo calculate the values of our condition $\left(u_0\right)_{i,j}$. The procedure we will use to solve the problem is an iterative method. Let us enumerate the iterates with the superscript k, $u^k_{i,j}$. Let us take $u^0_{i,j} = (u_0)_{i,j}$. In order to calculate the following iterates we will use the conditions of the problem, if

$$u = u_0, \text{ then } \frac{\sigma_p^2}{2} u_{xx} + \rho\sigma_p\sigma_q u_{xy} + \frac{\sigma_q^2}{2} u_{yy} - cu < 0.$$

Otherwise, if

$$\frac{\sigma_p^2}{2} u_{xx} + \rho\sigma_p\sigma_q u_{xy} + \frac{\sigma_q^2}{2} u_{yy} - cu = 0, \text{ then } u_0 - u < 0.$$

We will therefore use the iteration

$$u^{k+1}_{i,j} = \max\left(\frac{e_x\left(u^k_{i+1,j} + u^k_{i-1,j}\right) + e_y\left(u^k_{i,j+1} + u^k_{i,j-1}\right)}{2\left(c + e_x + e_y\right)} \right.$$

$$\left. + \frac{e_{xy}\left(u^k_{i+1,j+1} - u^k_{i+1,j-1} - u^k_{i-1,j+1} + u^k_{i-1,j-1}\right)}{2\left(c + e_x + e_y\right)}, (u_0)_{i,j} \right),$$

where we have defined

$$e_x = \frac{\sigma_p^2}{\mathrm{d}x^2}, \quad e_y = \frac{\sigma_q^2}{\mathrm{d}y^2}, \quad e_{xy} = \frac{\rho\sigma_p\sigma_q}{2\mathrm{d}x\mathrm{d}y}.$$

We will stop the iteration when the norm of the difference between two consecutive iterates is less than a known tollerance, $\left\|u^k - u^{k-1}\right\| < \varepsilon$. We will take $u = u^k$. This is an adaptation of the obstacle problem which is usually solved by using the iterative method *SOR*, which is a variation of the Gauss-Seidel method. Information can be found in [**10**] or [**17**]. For a more theoretic treatment of the *SOR* method one can check [**18**].

In order to show the results we will take a particular case using the same parameters as before. However, we will take $L_m = 150, M_m = 150$ and we obtain that $L_p = 40, M_p = 40$.

We now show the graphs corresponding to the two boundary conditions mentioned before, Fig. 5 and 6.

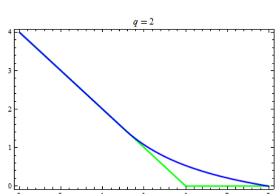

FIGURE 5. Curve $\mathcal{P}(p, 0)$ (blue) with initial condition $\Lambda(p, 0)$ (green).

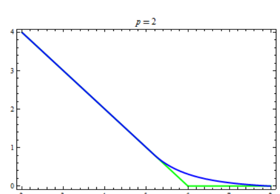

FIGURE 6. Curve $\mathcal{P}(0, q)$ (blue) with initial condition $\Lambda(0, q)$ (green).

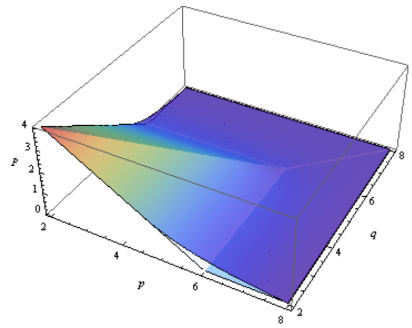

FIGURE 7. Surfaces corresponding to $\mathcal{P}(p,q)$ (colors) i $\Lambda(p,q)$ (white).

We now show the value of the option on the plane p,q relying on the initial condition, Fig. 7

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