The Euler-Lagrange Equation

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1 Introduction

The Euler-Lagrange equation is a powerful tool that appears in several fields such as theoretical physics and applied mathematics. Roughly speaking, it provides an equation for a function that minimizes an integral expression, namely, if we have

$$J(y) := \int_{\Omega} f(y)\mathrm{d}x,$$

the Euler-Lagrange equation gives a (differential) equation for the function $y(x)$ that minimizes $J(y)$. The sense and framework in which we will work, derive and understand the equation will be defined later on. In the following report we will derive the equation and we will give some applications of it.

2 Derivation

Even though that there are much more general mathematical frameworks in which one can derive optimality conditions for minimization problems, we will use the derivation provided in the variational calculus for functions of one variable and fix boundary.

Let us consider a function f continuously differentiable and a closed interval $[a, b] \subset \mathbb{R}$ with $a < b$. We will think of f as a functional depending on the variables $x, y(x), \dot{y}(x)$. Our aim is to find a function $y(x)$ that fulfills $y(a) = y_a$ and $y(b) = y_b$, with y_a, y_b given, which minimizes

$$J(y) := \int_a^b f(x, y, \dot{y}) \mathrm{d}x. \tag{1}$$

If y minimizes J subject to the boundary conditions, any slight perturbation of y that preserves the boundary must increase J.

Let $g_\alpha := y(x) + \alpha\eta(x)$ be such a perturbation, where α is meant to be small and η a continuously differentiable function fulfilling $\eta(a) = \eta(b) = 0$. Then define

$$f_\alpha := f(x, g_\alpha, \dot{g}_\alpha),$$

$$J_\alpha = \int_a^b f_\alpha \mathrm{d}x.$$

The derivative of J_α with respect to α is given by

$$\frac{\mathrm{d}J_\alpha}{\mathrm{d}\alpha} = \frac{\mathrm{d}}{\mathrm{d}\alpha} \int_a^b f_\alpha \mathrm{d}x = \int_a^b \frac{\mathrm{d}f_\alpha}{\mathrm{d}\alpha} \mathrm{d}x$$

and

$$\frac{\mathrm{d}f_\alpha}{\mathrm{d}\alpha} = \eta \frac{\partial f_\alpha}{\partial g_\alpha} + \dot{\eta} \frac{\partial f_\alpha}{\partial \dot{g}_\alpha},$$

so

$$\frac{\mathrm{d}J_\alpha}{\mathrm{d}\alpha} = \int_a^b \eta \frac{\partial f_\alpha}{\partial g_\alpha} + \dot{\eta} \frac{\partial f_\alpha}{\partial \dot{g}_\alpha} \mathrm{d}x.$$

When $\alpha = 0$ we have that $g_\alpha = y$, $f_\alpha = f$ and J_α has a minimum, so that

$$\left. \frac{\mathrm{d}J_\alpha}{\mathrm{d}\alpha} \right|_{\alpha=0} = \int_a^b \eta \frac{\partial f}{\partial y} + \dot{\eta} \frac{\partial f}{\partial \dot{y}} \mathrm{d}x = 0.$$

Integrating by parts the term with $\dot{\eta}$ we obtain

$$\left. \frac{\mathrm{d}J_\alpha}{\mathrm{d}\alpha} \right|_{\alpha=0} = \int_a^b \left(\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial \dot{y}} \right) \eta \mathrm{d}x + \left(\eta \frac{\partial f}{\partial \dot{y}} \right)_a^b = 0.$$

Since $\eta(a) = \eta(b) = 0$ we get

$$\left. \frac{\mathrm{d}J_\alpha}{\mathrm{d}\alpha} \right|_{\alpha=0} = \int_a^b \left(\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial \dot{y}} \right) \eta \mathrm{d}x = 0.$$

Applying the Fundamental Lemma of the Variational Calculus we obtain the so-called Euler-Lagrange equation

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial \dot{y}} = 0. \tag{2}$$

A more general expression can be provided, in case that f depends on further derivatives of y, $y^{(k)}$, for $k = 1, \ldots, n$, we have

$$\frac{\partial f}{\partial y} + \sum_{k=1}^{n} (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}x^k} \frac{\partial f}{\partial y^{(k)}} = 0, \tag{3}$$

where the factor $(-1)^k$ comes from an iterative integration by parts. If f instead depends on several functions of one variable y_j, (3) needs to be fulfilled for each y_j, that is,

$$\frac{\partial f}{\partial y_j} + \sum_{k=1}^{n} (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}x^k} \frac{\partial f}{\partial y_j^{(k)}} = 0.$$

Another expression we will use later on is the case in which there are more than one independent variables involved, x_i. In this case and for f depending on the first derivatives of y, we have that

$$\frac{\partial f}{\partial y} - \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \frac{\partial f}{\partial(\partial_i y)} = 0, \tag{4}$$

where

$$\partial_i y := \frac{\partial y}{\partial x_i}.$$

Before going to the applications, we will give a result that will make our calculations easier. In the case that f does not depend on the independent variable x, this is,

$$\frac{\partial f}{\partial x} = 0,$$

(2) can be integrated obtaining

$$f - \dot{y}\frac{\partial f}{\partial \dot{y}} = C, \tag{5}$$

where C is a constant. This is known as the Beltrami identity.

3 Applications

In this section we will use the former equations in order to provide a few applications of the Euler-Lagrange equation. The first one will be about physics and two of them will be purely mathematical related to differential manifolds.

3.1 Spring-pendulum

In theoretical mechanics, the Lagrangian of a classical system is a functional that contains the physical information of the system. According to the *Hamilton's Principle*, the trajectory $q(t)$ described by a particle is the one that minimizes the *action*

$$S(q_j) := \int_{t_1}^{t_2} L(t, q_j, \dot{q}_j)\mathrm{d}t,$$

where $L(t, q_j, \dot{q}_j)$ is the Langrangian and q_j for $j = 1, \ldots, m$ the coordinates describing the system. Under the assumption of having a kinetic energy depending on ${\dot{q}_j}^2$ and that the forces acting on the system are derived from a time-independent potential field, namely, $F(q_j) = -\nabla V(q_j)$, the Lagrangian can be written as

$$L := T - V, \tag{6}$$

that is, the difference between kinetic and potential energy and it needs to fulfill the Euler-Lagrange equation according to the Hamilton's Principle. In this example we are going to derive the differential equations describing a pendulum connected to a fix point by a spring. The pendulum is assumed to have mass m and the spring constant is k. The system is under the gravitational field of intensity g.

The advantage of using the Lagrangian formalism to describe the system is that we can introduce the *generalized coordinates*, this means, that we do not necessarily have to work with Cartesian coordinates, but we can use the ones that describe easily/best our system. Afterward, we apply (2) to L for each of the q_j. In our case, we will take $q_1 = r$ and $q_2 = \varphi$ and we will describe our system on the plane by using polar coordinates

$$\begin{cases} x(t) := r(t)\cos(\varphi(t)), \\ y(t) := r(t)\sin(\varphi(t)). \end{cases}$$

The velocity of the system is the time derivative of the position

$$\begin{cases} \dot{x} = \dot{r}\cos(\varphi) - r\dot{\varphi}\sin(\varphi), \\ \dot{y} = \dot{r}\sin(\varphi) + r\dot{\varphi}\cos(\varphi). \end{cases}$$

Now we compute the kinetic energy of the system

$$T := \frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right).$$

The gravitational force is given by the field $F_G = (0, -mg)^T$, which comes from the potential

$$V_G = mgy = mgr\sin(\varphi),$$

recalling that $F = -\nabla V$. Now we need to find a potential for the elastic force, the one produced by the spring. From the *Hooke's Law* we know that this force is given by $F_H = -k(x, y)^T$, which admits the potential

$$V_H = \frac{1}{2}k(x^2 + y^2) = \frac{1}{2}kr^2.$$

Thus, we can write the Lagrangian (6) as

$$L(r, \varphi, \dot{r}, \dot{\varphi}) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - mgr\sin(\varphi) - \frac{1}{2}kr^2. \tag{7}$$

We need now to apply (2) to L using r, φ. For r we obtain

$$\frac{\partial L}{\partial r} = mr\dot{\varphi}^2 - mg\sin(\varphi) - kr,$$
$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}.$$

For φ we have

$$\frac{\partial L}{\partial \varphi} = -mgr\cos(\varphi),$$
$$\frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi}.$$

By using the former derivatives we can write (2) for r, φ,

$$\begin{cases} \ddot{r} = r\dot{\varphi}^2 - g\sin(\varphi) - \dfrac{k}{m}r, \\ \\ \dfrac{\mathrm{d}\left(r^2\dot{\varphi}\right)}{\mathrm{d}t} = -gr\cos(\varphi). \end{cases} \tag{8}$$

Since we are dealing with a second order system of two differential equations, we require of two initial conditions for each coordinate, namely, initial position and initial velocity, $r(0) = r_0, \dot{r}(0) = v_0, \varphi(0) = \varphi_0, \dot{\varphi}(0) = \omega_0$.

3.2 Geodesics

Another well-known application of the variational calculus is a way of finding the differential equations for the geodesics of a manifold, i.e. the path on the manifold that minimizes the distance between two points. We are going to derive the equations for a curve on a sphere of radius R.

The length of arc of a curve $\gamma : [0, 1] \to M$ on a Riemannian manifold M with metric g is given by

$$\mathcal{L} = \int_0^1 \sqrt{g(\dot{\gamma}(\tau), \dot{\gamma}(\tau))} \mathrm{d}\tau. \tag{9}$$

If we consider the parametrization

$$S(\varphi, \theta) = (R\cos(\varphi)\sin(\theta), R\sin(\varphi)\sin(\theta), \cos(\theta)), \quad (\varphi, \theta) \in [0, 2\pi] \times [0, \pi)$$

for the sphere, the metric g is given by

$$g = \begin{pmatrix} R^2 \sin^2(\theta) & 0 \\ 0 & R^2 \end{pmatrix}$$

Thus, (9) becomes

$$\mathcal{L} = \int_0^1 R \left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2(\theta)\right)^{1/2} \mathrm{d}\tau. \tag{10}$$

Now we proceed to apply (2) for the functional $f := R\left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2(\theta)\right)^{1/2}$,

$$\frac{\partial f}{\partial \varphi} = 0,$$

$$\frac{\partial f}{\partial \dot{\varphi}} = \frac{R\sin^2(\theta)\dot{\varphi}}{\left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2(\theta)\right)^{1/2}}.$$

If we parametrize the curve by using the arc of length,

$$s(t) := \int_0^t R\left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2(\theta)\right)^{1/2} \mathrm{d}\tau,$$

we have that,

$$\frac{\mathrm{d}s}{\mathrm{d}t} = R\left(\dot{\theta}^2 + \dot{\varphi}^2\sin^2(\theta)\right)^{1/2}, \tag{11}$$

and we can therefore rewrite the derivative of f with respect $\dot{\varphi}$ as

$$\frac{\partial f}{\partial \dot{\varphi}} = \frac{R\sin^2(\theta)\dot{\varphi}}{\left(\dot{\theta}^2 + \dot{\varphi}^2\sin^2(\theta)\right)^{1/2}} = R^2\sin^2(\theta)\frac{\mathrm{d}\varphi}{\mathrm{d}s}.$$

Similarly,

$$\frac{\partial f}{\partial \theta} = \frac{R\sin(\theta)\cos(\theta)\dot{\varphi}^2}{\left(\dot{\theta}^2 + \dot{\varphi}^2\sin^2(\theta)\right)^{1/2}} = R^2\sin(\theta)\cos(\theta)\left(\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right)^2\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)$$

$$\frac{\partial f}{\partial \dot{\theta}} = \frac{R\dot{\theta}}{\left(\dot{\theta}^2 + \dot{\varphi}^2\sin^2(\theta)\right)^{1/2}} = R^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right).$$

Putting all together and changing from t to s in the Euler-Lagrange equation, we obtain the system of ODEs for φ, θ,

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s}\left(\sin(\theta)^2\dfrac{\mathrm{d}\varphi}{\mathrm{d}s}\right) = 0, \\[2ex] \dfrac{\mathrm{d}^2\theta}{\mathrm{d}s^2} - \sin(\theta)\cos(\theta)\left(\dfrac{\mathrm{d}\varphi}{\mathrm{d}s}\right)^2 = 0. \end{cases} \tag{12}$$

3.3 Surface

In the former section we investigated the minimization of the length of arc on a manifold. Now we are going to look for the manifold that minimizes the surface. If we assume that we have a differential manifold $M \subset \mathbb{R}$ and we further assume that we have the parametrization $z : U \times V \to \mathbb{R}$, $U, V \subset \mathbb{R}$ open, such that M can be expressed by components as $(x, y, z(x, y))$, then its surface is given by

$$S := \int_U \int_V \left(1 + z_x^2 + z_y^2\right)^{1/2} \mathrm{d}x\mathrm{d}y, \tag{13}$$

where

$$z_x := \frac{\partial z}{\partial x}, \qquad z_y := \frac{\partial z}{\partial y}.$$

In this case we need to apply (4) to the functional $f := \left(1 + z_x^2 + z_y^2\right)^{1/2}$. The first derivatives are simply given by

$$\frac{\partial f}{\partial z_x} = z_x \left(1 + z_x^2 + z_y^2\right)^{-1/2},$$

$$\frac{\partial f}{\partial z_y} = z_y \left(1 + z_x^2 + z_y^2\right)^{-1/2}.$$

Now taking partial derivatives of both terms above with respect x, y respectively, (4) reads

$$\left(1 + z_x^2 + z_y^2\right)\left(z_{xx} + z_{yy}\right) + z_x z_{xx} + \left(z_x + z_y\right) z_{xy} + z_y z_{yy} = 0 \tag{14}$$

A numerical solution of (14) by using the finite difference method and taking catenary arches on the boundary.