Geodesics on an ellipsoid

Marc Puche

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Contents

1 Introduction

The term geodesic refers in mathematics to the curve on a surface that minimizes the length of arc between two points. A well-known application is the field of *Geodesy*, which is the study of such curves on the Earth as well as its gravitational field properties. In this report we are going to give a mathematical model for the Earth and derive the equations of a geodesic on its surface. Finally we will calculate distances between pairs of points and introduce the the so-called *Two geodetic problems*.

2 Derivation

The first step we will do is to give a mathematical model for the Earth. The first thing one may come up with is, of course, a sphere. However, the Earth is clearly not a sphere, since it's flatter in the poles than in the equator. Taking this simple fact into account, we can think of a better approach, namely, and ellipsoid of revolution, which is the 3-dimensional shape of an ellipse on the xz-plane spun around the z-axis.

A parametrization of the resulting surface is

$$f(\varphi, \theta) = (a\cos(\varphi)\sin(\theta), a\sin(\varphi)\sin(\theta), b\cos(\theta)), \qquad (\varphi, \theta) \in [0, 2\pi) \times [0, \pi],$$

where a, b are the semi-axis of the ellipsoid, being $0 < b < a$. The ellipsoid is a Riemannian manifold whose induced metric is component-wise given by

$$g_{\varphi\varphi} = \left(\frac{\partial f}{\partial \varphi}, \frac{\partial f}{\partial \varphi}\right), \qquad g_{\theta\varphi} = \left(\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \varphi}\right),$$

$$g_{\varphi\theta} = \left(\frac{\partial f}{\partial \varphi}, \frac{\partial f}{\partial \theta}\right), \qquad g_{\theta\theta} = \left(\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \theta}\right),$$

where $(\cdot, \cdot)$ denotes the inner product in $\mathbb{R}^3$. If we take the partial derivatives of f with respect the angles φ and θ,

$$\frac{\partial f}{\partial \varphi} = (-a\sin(\varphi)\sin(\theta), a\cos(\varphi)\sin(\theta), 0),$$

$$\frac{\partial f}{\partial \theta} = (a\cos(\varphi)\cos(\theta), a\sin(\varphi)\cos(\theta), -b\sin(\theta)),$$

then the metric is

$$g = \begin{pmatrix} a^2\sin^2(\theta) & 0 \\ 0 & a^2\cos^2(\theta) + b^2\sin^2(\theta) \end{pmatrix}$$

Using the Einstein's convention, the equations of the geodesic on a Riemannian manifold are given by

$$\ddot{x}^k + \Gamma^k_{ij}\dot{x}^i\dot{x}^j = 0,$$

where

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}\left(\frac{\partial g_{li}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right)$$

are the Christoffel symbols and g^{kl} denotes (by components) the inverse matrix of g. The non-vanishing Christoffel symbols are the following ones

$$\Gamma^{\varphi}_{\varphi\theta} = \Gamma^{\varphi}_{\theta\varphi} = \cot(\theta),$$

$$\Gamma^{\theta}_{\theta\theta} = -\frac{a^2 - b^2}{a^2 + b^2 \tan^2(\theta)} \tan(\theta),$$

$$\Gamma^{\theta}_{\varphi\varphi} = -\frac{a^2}{a^2 + b^2 \tan^2(\theta)} \tan(\theta).$$

Then we obtain the two differential equations for φ, θ,

$$\begin{cases} \ddot{\varphi} + 2\cot(\theta)\dot{\theta}\dot{\varphi} = 0, \\ \ddot{\theta} - \dfrac{a^2 \tan(\theta)}{a^2 + b^2 \tan^2(\theta)}\dot{\varphi}^2 - \dfrac{(a^2 - b^2)\tan(\theta)}{a^2 + b^2 \tan^2(\theta)}\dot{\theta}^2 = 0. \end{cases} \tag{1}$$

The former system can not be solved in the sense of expressing φ, θ in terms of elemental functions. However, the first equation can be easily integrated. If we assume that we have initial conditions $\varphi(0) = \varphi_0, \theta(0) = \theta_0, \dot{\varphi}(0) = \omega, \dot{\theta}(0) = \Omega$. By using these known constants, we can integrate the first equation in (1) and obtain

$$\begin{cases} \dot{\varphi} = \omega \dfrac{\sin^2(\theta_0)}{\sin^2(\theta)}, \\ \ddot{\theta} - \dfrac{a^2 \tan(\theta)}{a^2 + b^2 \tan^2(\theta)}\dot{\varphi}^2 - \dfrac{(a^2 - b^2)\tan(\theta)}{a^2 + b^2 \tan^2(\theta)}\dot{\theta}^2 = 0. \end{cases} \tag{2}$$

3 The two geodetic problems

As we have previously explained, we are going to calculate distances between pairs of points on the surface. Let us name these two points x_A, x_B and assume that we have a geodesic curve γ on the manifold such that $\gamma(0) = x_A$ and $\gamma(T) = x_B$. This curve can be expressed as

$$\gamma(s) = \left(a\cos(\varphi(s))\sin(\theta(s)), a\sin(\varphi(s))\sin(\theta(s)), b\cos(\theta(s))\right),$$

where $\theta(s), \varphi(s)$ fulfill (1). The distance between these two points is calculated by finding the length of the geodesic connecting x_A, x_B, this is,

$$\mathcal{L} = \int_0^T \left(a^2\dot{\varphi}^2\sin^2(\theta) + \dot{\theta}^2\left(a^2\cos^2(\theta) + b^2\sin^2(\theta)\right)\right)^{1/2} \mathrm{d}s.$$

Now, a natural question arises. Given the initial point, direction and length of arc, where is the second point? In other words, if we have $\theta_0, \varphi_0, \omega, \Omega$ and $\mathcal{L}$, can we find T and therefore $\theta(T), \varphi(T)$ and the point $\gamma(T)$? This question is known as *The First (Direct) Geodetic Problem*. The problem can be also posed the other way around, as we have previously done it. Given two points, determine the direction and length of arc connecting them. This is know as *The Second (Inverse) Geodetic Problem*. For this one, we particularly know $\gamma(0)$ and $\gamma(T)$, and without loss of generality we can set $T = 1$.

The two problems are clearly equivalent on the plane. However, proving their equivalent in general may be complicated or actually not even true, since the geodesics are defined locally on the manifolds, and global extensions may not even exist. In our case, everything needs to be solved numerically, but the equivalence can be actually found.

Before closing this section, a brief explanation of why Ω, ω give the direction from the starting point. The tangent vector at the starting point is nothing else but $\dot{\gamma}(0)$. Since $\gamma(s) = f(\varphi(s), \theta(s))$, we can use the chain rule to calculate $\dot{\gamma}$,

$$\dot{\gamma} = \frac{\partial f}{\partial \varphi}\dot{\varphi} + \frac{\partial f}{\partial \theta}\dot{\theta}.$$

On our manifold, the basis of the tangent space at each point is in fact given by

$$e_\varphi = \frac{\partial f}{\partial \varphi}, \qquad e_\theta = \frac{\partial f}{\partial \theta},$$

so that

$$\dot{\gamma} = \dot{\varphi}e_\varphi + \dot{\theta}e_\theta.$$

At $s = 0$ we have

$$\dot{\gamma}(0) = \dot{\varphi}(0) e_{\varphi}(\varphi(0), \theta(0)) + \dot{\theta}(0) e_{\theta}(\varphi(0), \theta(0)),$$

which by definition of the constants is

$$\dot{\gamma}(0) = \omega e_{\varphi}(\varphi_0, \theta_0) + \Omega e_{\theta}(\varphi_0, \theta_0).$$

So the direction of the curve at the starting point can be expressed on the basis of the tangent space (at the starting point) as

$$d = \begin{pmatrix} \omega \\ \Omega \end{pmatrix}$$

4 Distance Barcelona-Hamburg

Finally, we are going to calculate the distance between two cities.

The coordinates of the cities are

City	θ (rad)	φ (rad)
Barcelona	0.8485	0.0381
Hamburg	0.6359	0.1746

This gives a length

$$\mathcal{L} = 1198.916\ km,$$

and direction

$$\omega = 0.3146\ rad, \qquad \Omega = -0.05187\ rad.$$